Z^4-connectivity of Graphs Satisfying Chvatal-condition

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Abstract: In this paper, a simple graph G with the order at least 3 satisfying Chvatal-condition is A-connected if and only if G is not the 4-cycle was showed, in which A is an abelian group of order at least 4.

1. Introduction

Graphs in this paper are finite and may have multiple edges. A graph is simple if it has no multiple edges and loops. Let G be a graph with an orientation D. We use $V(G)$ to denote the set of vertexes of G and $E(G)$ to denote the set of edges of G. For a vertex $v \in V(G)$, we use $E^+(v)$ (or $E^-(v)$, respectively) to denote the set of edges with tails (or heads, respectively) at v. For subgraphs V_1 and V_2 of G, $e(V_1,V_2)$ denotes the number of edges with one end in V_1 and the other end in V_2.

Let A be an (additive) abelian group with identity 0 and $A^* = A - \{0\}$, let $F(G,A)$ denote the set of all functions from $E(G)$ to A, and $F^*(G,A)$ denote the set of all functions from $E(G)$ to A^*. Given a function $f \in F(G,A)$, let $\partial f : V(G) \to A$ be given by $\partial f(v) = \sum_{e\in E^+(v)} f(e) - \sum_{e\in E^-(v)} f(e)$, where ``$\sum$" refers to the addition in A.

Let Z_k be an (additive) abelian group of k elements with identity 0. $f \in F(G,Z_k)$ is called a Z_k-flow in G if $\partial f(v) = 0$ for each $v \in V(G)$. For an edge $e \in E(G)$, we call $f(e)$ the flow value of e. The support of f is defined by $S(f) = \{e \in E(G) | f(e) \neq 0\}$. f is called a nowhere-zero Z_k-flow if $S(f) = E(G)$. For an integer $k \geq 2$, a nowhere-zero k-flow of G is an integer-valued function f on $E(G)$ such that $0 < |f(e)| < k$ for each $e \in E(G)$, and $\partial f(v) = 0$ for each $v \in V(G)$. It is well known that G has a nowhere-zero Z_k-flow if and only if it has a nowhere-zero k-flow. Therefore, a Z_k-flow is also called a k-flow.

The concept of group connectivity was introduced by Jaeger et al. [2] as a generalization of nowhere-zero flows. For a graph G, a function $b : V(G) \to A$ is called an A-valued zero-sum

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function on G if $\sum_{v\in V(G)} b(v)=0$.The set of all A-valued zero-sum functions on G is denoted by $Z(G,A)$. Given $b\in Z(G,A)$ and an orientation D of G, a function $f\in F^{*}(G,A)$ is an (A,b)-nowhere-zero flow if $\partial f=b$.A graph G is A-connected if G has an orientation D such that for any $b\in Z(G,A)$, G has an (A,b)-nowhere-zero flow. As noted in [2], G is A-connected or not is independent of the choice of the orientation D.

In [3], Jaeger et al. made the following conjecture on Z_3-connectivity, which implies Tutte's well-known 3-flow conjecture.

Conjecture 1.1[3] Every 5-edge connected graph is Z_3-connected.

If G is a simple graph on n vertices, the degree sequence of G is denoted by $\pi(G)=(d_1,d_2,\cdots d_n)$, where $d_1\le d_2\le\cdots\le d_n$. If G satisfies $Chva'tal$-condition that $d_m\ge m+1$ or $d_{n-m}\ge n-m$ for $1\le m<\frac{n}{2}$, we have the following known results.

(1) If G satisfies $p\acute{o}sa$-condition, then G satisfies $Chva'tal$-condition, but the opposite is not always true.

(2) If G satisfies $Chva'tal$-condition, then G is Hamiltonian, and G has a nowhere-zero 4-flow, but G is not always Z_4- connected.

Let A be an abelian group of order at least 4. In [5], Yue Zhang et al. characterized a simple graph G satisfying $p\acute{o}sa$-condition with $|V(G)|\ge 3$ is A-connected if and only if $G\ne C_4$. In this paper, we will further characterize a simple graph G satisfying $Chva'tal$-condition with $|V(G)|\ge 3$ is A-connected if and only if $G\ne C_4$(Theorem 1).

2. Preliminaries

Let G be a graph. For a subset $X\subseteq E(G)$, the contraction G/X is the graph obtained from G by identifying the two ends of each edge e in X and deleting e. For convenience, we write G/e for $G/\{e\}$, where $e\in E(G)$. If H is a subgraph of G, then we write G/H for $G/E(H)$.

In order to prove Theorem 1, we need the following known results and useful lemmas.

Lemma 2.1[4] let A be an abelian group, then we have the following results:

(1) If H is a subgraph of G and if both H and G/H are A-connected, then G is A-connected.

(2) C_n is A-connected if and only if $|A|\ge n+1$, where C_n denotes the n-cycle.

(3) K_1 is A-connected.

Lemma 2.2[1] Let A be an abelian group, if $|A|\ge 4$ and $n\ge m\ge 3$, then $K_{m,n}$ is A-connected, where $K_{m,n}$ denotes the complete bipartite graph.

In the following we always assume that G is a 2-edge-connected graph, $V(G)=\{v_1,v_2,\cdots v_n\}$ and $d_G(v_i)=d_i\ (i=1,2,\cdots n)$, where $d_1 \le d_2 \le \cdots \le d_n$.

Lemma 2.3 Let $|G|=n\ge 4$, G satisfies $Chva'tal$-condition and $G\ne C_4$.Then G contains a triangle with a vertex having degree at least $\left\lceil \frac{n}{2} \right\rceil$ or G is a bipartite graph of order $\frac{n}{2}\times\frac{n}{2}$ which has a subgraph $K_{3,\frac{n}{2}}\ (n\ge 6)$.

Proof. We consider the following two cases.

Case 1 n is odd.

By $Chva'tal$-condition, we have $d_{\frac{n-1}{2}} \ge \frac{n+1}{2}$ or $d_{\frac{n+1}{2}} \ge \frac{n+1}{2}$, then there exists an edge $v_{\frac{n+1}{2}}v_i \in E(G)$ with $i > \frac{n+1}{2}$ and $d_i \ge \frac{n+1}{2}$. Since $v_{\frac{n+1}{2}}$ and v_i having at least one same adjacent vertex, then G contains a triangle with a vertex having degree at least $\left\lceil \frac{n}{2} \right\rceil$.

Case 2 n is even.

By $Chva'tal$-condition, we have $d_{\frac{n}{2}-1} \ge \frac{n}{2}$ or $d_{\frac{n}{2}+1} \ge \frac{n}{2}+1$. If $d_{\frac{n}{2}-1} \ge \frac{n}{2}$, then there exists an edge $v_{\frac{n}{2}-1}v_i \in E(G)$ with $i > \frac{n}{2}-1$ and $d_i \ge \frac{n}{2}$. If $N_G(v_{\frac{n}{2}-1})$ or $N_G(v_i)$ is not an independent set, then G contains a triangle with a vertex having degree at least $\left\lceil \frac{n}{2} \right\rceil$. If $N_G(v_{\frac{n}{2}-1})\cap N_G(v_i)\ne\phi$, then G also contains a triangle with a vertex having degree at least $\left\lceil \frac{n}{2} \right\rceil$. Thus, we only need to consider the case when $N_G(v_{\frac{n}{2}-1})$ and $N_G(v_i)$ are independent set, and $N_G(v_{\frac{n}{2}-1})\cap N_G(v_i)=\phi$. In this case, we have $\left|N_G(v_{\frac{n}{2}-1})\right| = |N_G(v_i)| = \frac{n}{2}$ and $N_G(v_{\frac{n}{2}-1})\cup N_G(v_i)=V(G)$. Then G is a bipartite graph which has the bipartition $(N_G(v_{\frac{n}{2}-1}), N_G(v_i))$.It is easy to see that there are at least $\frac{n}{2}+2$ vertices having degree at least $\frac{n}{2}$. If $n=4$, then $G=C_4$, a contradiction. If $n\ge 6$, then there are at least 3 vertices having degree at least $\frac{n}{2}$ in $N_G(v_{\frac{n}{2}-1})$ or $N_G(v_i)$. Thus, G is a bipartite graph of order $\frac{n}{2}\times\frac{n}{2}$ which has a subgraph $K_{3,\frac{n}{2}}$ $(n\geq 6)$. If $d_{\frac{n}{2}+1}\geq\frac{n}{2}+1$, then there exists an edge $v_{\frac{n}{2}+1}v_i\in E(G)$ where $i>\frac{n}{2}+1$, $d_i\geq\frac{n}{2}+1$, $v_{\frac{n+1}{2}}$ and v_i having at least one same adjacent vertex. Therefore, G contains a triangle with a vertex having degree at least $\left\lceil\frac{n}{2}\right\rceil$.

3. Conclusions

Theorem 1 Let $|G|=n\geq 3$. If G satisfies $Chva'tal$-condition, then G is A-connected if and only if $G\neq C_4$, where A is an abelian group of order at least 4 and C_4 is a cycle of length 4.

Proof. If G is A-connected, then by Lemma 2.1(2), we have $G\neq C_4$. Conversely, if $G\neq C_4$, then we will prove that G is A-connected.

If $n=3$, it is easy to see that G is a triangle. By Lemma 2.1(2), G is A-connected. If $n\geq 4$, we consider the following two cases.

Case 1 G contains a triangle.

By Lemma 2.3, G contains a triangle with a vertex v' having degree at least $\left\lceil\frac{n}{2}\right\rceil$. In G, contract this triangle to a vertex u^*, we get a new graph G^1. In G^1, contract all 2-cycles, loops and the triangles which contain the vertex u^*, we get a new graph G^2. For convenience, we still let u^* to denote the new vertex which contracted into. Recursively contract all 2-cycles, loops and the triangles which contain the vertex u^*, eventually, we get the simple graph G^t. Clearly, there is a connected subgraph H of G such that $G^t=G/H$. Let $H'=G^t-u^*$, $|V(H)|=n_1$ and $|V(H')|=n_2$. Clearly, $n_1\geq 3$ and $n_2\geq 0$. If $n_2>0$, since G^t is a simple graph, we have that $e(v,H)\leq 1$ for any $v\in V(H')$. Let $u_1,u_2,\cdots u_r$ be all vertices for which $e(u_i,H)=1$, then $\{u_1,u_2,\cdots u_r\}$ is a independent set.

First we consider the case when n is even. If $n_1>n_2$, then $n_2<\frac{n}{2}$. By $Chva'tal$-condition, we have $d_{n_2}\geq n_2+1$ or $d_{n-n_2}\geq n-n_2$, then $d_{n_2}\geq n_2+1$ or $d_{n_1}\geq n_1$. Since $d_G(v)\leq n_2$ for any $v\in V(H')$, then $d_{n_2}\leq n_2$, a contradiction to $d_{n_2}\geq n_2+1$. Therefore, $d_{n_1}\geq n_1$, then there are at least n_2+1 vertices having degree at least n_1 in G, then it is easy to see that there are at least n_2+1 vertices having degree at least n_1 in H. Therefore, $r\geq n_2+1$, a contradiction. If $n_1\leq n_2$, since $v'\in V(H)$ and $d_G(v')\geq\frac{n}{2}$, then $r\geq\frac{n}{2}-n_1+1\geq 1$

and $d_G(u_i) \le n_2 - r + 1 \le n_2 - (\frac{n}{2} - n_1 + 1) + 1 = \frac{n}{2}$ $(i = 1,2,\cdots r)$. By $Chva'tal$ -condition, we have $d_{\frac{n}{2}-1} \ge \frac{n}{2}$ or $d_{\frac{n}{2}+1} \ge \frac{n}{2} + 1$. If $d_{\frac{n}{2}-1} \ge \frac{n}{2}$ and $d_G(u_i) < \frac{n}{2}$ for any u_i, then there are at least $\frac{n}{2} + 2 - (n_2 - r) \ge \frac{n}{2} + 2 - n_2 + (\frac{n}{2} - n_1 + 1) = 3$ vertices having degree at least $\frac{n}{2}$ in H, we have $r \ge 3(\frac{n}{2} - n_1 + 1)$. Similarly, there are at least $\frac{n}{2} + 2 - (n_2 - r) \ge \frac{n}{2} + 2 - n_2 + 3(\frac{n}{2} - n_1 + 1) = n - 2n_1 + 5 \ge 5$ vertices having degree at least $\frac{n}{2}$ in H, we have $r \ge 5(\frac{n}{2} - n_1 + 1)$. Recursively we have $r > n_2$, a contradiction. If $d_{\frac{n}{2}-1} \ge \frac{n}{2}$ and $d_G(u_i) = \frac{n}{2}$ for one u_i, then there must be have $r = \frac{n}{2} - n_1 + 1$, $d_G(v') = \frac{n}{2}$ and v' is the only vertex which satisfies $e(v', H') > 0$ in H, a contradiction to G is Hamiltonian. If $d_{\frac{n}{2}+1} \ge \frac{n}{2} + 1$, then there are at least $\frac{n}{2} - (n_2 - r) \ge \frac{n}{2} - n_2 + (\frac{n}{2} - n_1 + 1) = 1$ vertex having degree at least $\frac{n}{2} + 1$ in H, we have $r \ge \frac{n}{2} + 1 - n_1 + 1$. Similarly, there are at least $\frac{n}{2} - (n_2 - r) \ge \frac{n}{2} - n_2 + (\frac{n}{2} + 1 - n_1 + 1) = 2$ vertices having degree at least $\frac{n}{2} + 1$ in H, we have $r \ge 2(\frac{n}{2} + 1 - n_1 + 1)$. Recursively we have $r > n_2$, a contradiction.

Now we consider the case when n is odd. If $n_1 \le \frac{n+1}{2}$, since $v' \in V(H)$ and $d_G(v') \ge \frac{n+1}{2}$, then $r \ge \frac{n+1}{2} - n_1 + 1 \ge 1$ and $d_G(u_i) \le n_2 - r + 1 \le n_2 - (\frac{n+1}{2} - n_1 + 1) + 1 = \frac{n-1}{2}$ $(i = 1,2,\cdots r)$. By $Chva'tal$ -condition, we have $d_{\frac{n-1}{2}} \ge \frac{n+1}{2}$ or $d_{\frac{n+1}{2}} \ge \frac{n+1}{2}$, then there are at least $\frac{n+1}{2}$ vertices having degree at least $\frac{n+1}{2}$ in G, then it is easy to see that there are at least $\frac{n+1}{2} - (n_2 - r) \ge \frac{n+1}{2} - n_2 + (\frac{n+1}{2} - n_1 + 1) = 2$ vertices having degree at least $\frac{n+1}{2}$ in H. Therefore, $r \ge 2(\frac{n+1}{2} - n_1 + 1)$. Similarly, there are at least $\frac{n+1}{2} - (n_2 - r) \ge \frac{n+1}{2} - n_2 + 2(\frac{n+1}{2} - n_1 + 1) \ge 3$ vertices having degree at least $\frac{n+1}{2}$ in H.

Therefore, $r \geq 3(\frac{n+1}{2} - n_1 + 1)$. Recursively we have $r > n_2$, a contradiction. If $n_1 > \frac{n+1}{2}$, then $n_2 < \frac{n-1}{2}$. By *Chva'tal*-condition, we have $d_{n_2} \geq n_2 + 1$ or $d_{n-n_2} \geq n - n_2$, then $d_{n_2} \geq n_2 + 1$ or $d_{n_1} \geq n_1$. Since $d_G(v) \leq n_2$ for any $v \in V(H')$, then $d_{n_2} \leq n_2$, a contradiction to $d_{n_2} \geq n_2 + 1$. Therefore, $d_{n_1} \geq n_1$, then there are at least $n_2 + 1$ vertices having degree at least n_1 in G, then it is easy to see that there are at least $n_2 + 1$ vertices having degree at least n_1 in H. Therefore, $r \geq n_2 + 1$, a contradiction.

In summary, we always have $n_2 = 0$ for n is even or odd, then $H = G, G^t = G/H = K_1$. By Lemma 2.1(1) and Lemma 2.1(3), G is A-connected.

Case 2 G does not contain triangles.

By Lemma 2.3, G is a bipartite graph of order $\frac{n}{2} \times \frac{n}{2}$ which has a subgraph $K_{3,\frac{n}{2}}$ $(n \geq 6)$. Contract the $K_{3,\frac{n}{2}}$ and recursively contract all resulting 2-cycles and loops, eventually, we get K_1. By Lemma 2.1(3), G is A-connected.

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